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A VARIATIONAL FORMULATION FOR A THREE-DIMENSIONAL EXTENSIBLE MARINE PIPE TRANSPORTING FLUID

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ABSTRACT

The purpose of this paper is to develop a three-dimensional model formulation of an extensible marine pipe transporting fluid via a variational approach. The elastica theory of extensible rod and the kinematics theory of mass transported on the moving frame are used to obtain the model formulation. The three deformation descriptions referring to the Cartesian coordinate are considered. they are the total Lagrangian, the updated Lagrangian, and the Eulerian. By the principle of virtual work-energy, the Euler equations is derived and can be validated by the vectorial summation of forces and moments.

KEYWORDS: Three-dimensional pipes transporting fluid, Large displacements, Large strain, Extensible pipe, Variational formulation, Elastica

INTRODUCTION

In literature, there are a limited number of research on three-dimensional analysis of marine pipes/risers, for example. Doll and Mote (1976), Bernitsas (1982), Felippa and Chung (1981). Huang and Kang (1991), Kokarakis and Bernitsas (1987), and Chung et al. (1994 a&b, 1996). All of these studies considered the effect of axial deformation using small strain analysis. Unfortunately, for highly flexible pipes, this constraint is no longer applicable.

Recently, Chucheepsakul et al. (2003) proposed a model formulation, which signifies the effect of large axial deformation and fluid transportation. The elastica theory was used to develop the two-dimensional model. Numerical demonstrations are given in Chucheepsakul et al. (2001) and Chucheepsakul and Monprapussorn (2001). For a complete model, a three-dimensional formulation including the effect of torsion should be developed.

The objective of this paper is to develop the model formulation of marine pipe/riser experiencing large displacement and large deformation in three-dimensional space. The formulation is developed by variational approach based on the elastica theory and the work-energy principle. The strain energy of the pipe composes of strain energy due to large axial deformation, bending, and twisting. Large axial strain consideration is investigated in three deformation descriptors, namely the total Lagrangian, the updated Lagrangian, and the Eulerian. The apparent tension concept and the dynamic interactions between fluid and pipe are used to derive the external virtual works of the pipe. The variational formulation is validated by the vectorial formulation, which considers the equilibrium of forces and moments of a three-dimensional pipe/riser segment.

ASSUMPTION

The following assumptions are used to stipulate the present formulation. a) The material used in the pipe is linearly elastic. b) The pipe is initially straight and has no residual stress at the undeformed state. c) The pipe's cross sections remain circular after change of cross-sectional size due to the Poisson effect. d) Longitudinal strain is large, while the effect of shear strain is small and can be neglected, so that the Kirchhoff's rod theories are usable. e) Every cross section remains plane and remains perpendicular to the axis. f) Radial lines of the sections remain straight and radial as the cross section rotates about the axis. g) The internal and external fluids are inviscid, incompressible and irrotational. Their densities are uniform along arc-length of the riser. h) The internal flow is the one-dimensional plug laminar flow. i) Morison's equation is adopted for evaluating external hydrodynamic forces of external fluid. j) The effect of rotary inertia is negligible

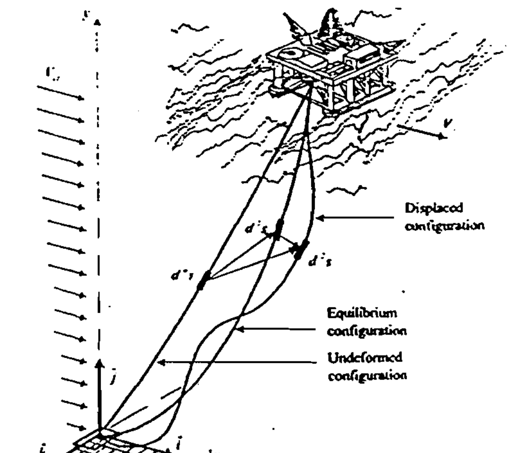

Figure 1. Three configurations of marine pipe.

MODELING AND PHYSICAL DESCRIPTIONS

The marine pipe is modeled to be three-dimensional rod with a ball joint at the bottom end and a slip joint at the top end. The three configurations of the pipe are depicted in Figure 1. At the undeformed configuration, the pipe is at rest and unstretched. Then, the pipe is subjected to the time-independent loads and its configuration changes to equilibrium configuration. Finally, at the displaced configuration, dynamic actions such as wave, unsteady current, and unsteady internal flow disturb the pipe to sustain vibration about the equilibrium configuration.

In this paper, three orthogonal coordinate systems are used to define position, motion, and deformation of an extensible marine pipe. The orthogonal triad system $\hat{t},\hat{n},\hat{b}$ and the cross-sectional principal axes system ${}^{i}x_1, {}^{i}x_2, {}^{i}x_3$ with unit normal vector ${}^{i}e_1, {}^{i}e_2, {}^{i}e_3$ are used as the local coordinate. The fixed Cartesian coordinate system ${}^{i}x, {}^{i}y, {}^{i}z$ with unit normal vector $\hat{i},\hat{j},\hat{k}$ is used as global coordinate. The left superscript represents the state of marine pipe where 0 represents the undeformed state, 1 represents the equilibrium state and 2 represents the displaced state. therefore, $i \in (0,1,2)$.

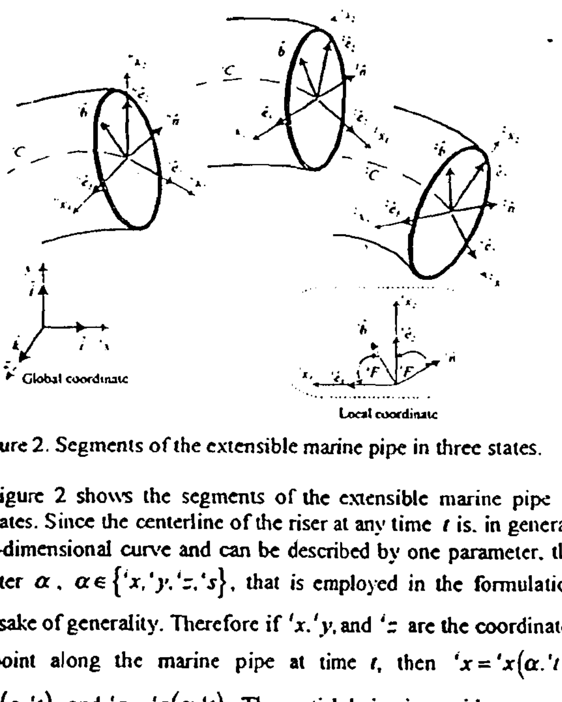

Figure 2. Segments of the extensible marine pipe in three states.

Figure 2 shows the segments of the extensible marine pipe in three states. Since the centerline of the riser at any time t is, in general, a three-dimensional curve and can be described by one parameter, the parameter α, $\alpha \in \{{}^{i}x, {}^{i}y, {}^{i}z, {}^{i}s\}$, that is employed in the formulation for the sake of generality. Therefore if ${}^{i}x, {}^{i}y$, and ${}^{i}z$ are the coordinates of a point along the marine pipe at time t, then ${}^{i}x = {}^{i}x(\alpha, {}^{i}t)$, ${}^{i}y = {}^{i}y(\alpha, {}^{i}t)$, and ${}^{i}z = {}^{i}z(\alpha, {}^{i}t)$. The partial derivatives with respect to α and time ${}^{i}t$ are represented by superscripts $()'$ and $(\dot{})$ respectively.

MEASUREMENTS OF AXIAL STRAINS

In Cartesian coordinate, the relations of differential arc-length at the undeformed state, the equilibrium state and the displaced state $({}^{0}s', {}^{1}s' \text{ and } {}^{2}s')$ can be expressed as

$$ {}^{0}s' = \sqrt{{}^{0}x'^2 + {}^{0}y'^2 + {}^{0}z'^2} $$

$$ {}^{1}s' = \sqrt{{}^{1}x'^2 + {}^{1}y'^2 + {}^{1}z'^2} = \sqrt{({}^{0}x' + {}^{1}u')^2 + ({}^{0}y' + {}^{1}v')^2 + ({}^{0}z' + {}^{1}w')^2} $$

$$ {}^{2}s' = \sqrt{{}^{2}x'^2 + {}^{2}y'^2 + {}^{2}z'^2} = \sqrt{({}^{0}x' + {}^{2}u')^2 + ({}^{0}y' + {}^{2}v')^2 + ({}^{0}z' + {}^{2}w')^2} $$

$$ {}^{2}s' = \sqrt{({}^{0}x' + {}^{1}u' + u')^2 + ({}^{0}y' + {}^{1}v' + v')^2 + ({}^{0}z' + {}^{1}w' + w')^2} \quad \text{(1 a-d)} $$

According to the mechanics of a deformable body, the definition of axial strain can be provided in three forms, namely the Total Lagrangian Descriptor, the Updated Lagrangian Descriptor, and the Eulerian Descriptor. Each of these forms can be demonstrated as follows.

Total Lagrangian Descriptor (TLD)

The coordinate that follows motion and deformation of a deformable body with respect to position, direction, and size of the body at the original state (or undeformed state herein) is said to be the total Lagrangian descriptor.

Total strain $\quad {}^{2}\bar{\varepsilon} = \dfrac{d^{2}s - d^{0}s}{d^{0}s} = \dfrac{d^{2}s}{d^{0}s} - 1 = \sqrt{1 + 2({}^{2}L)} - 1$

Static strain $\quad {}^{1}\bar{\varepsilon} = \dfrac{d^{1}s - d^{0}s}{d^{0}s} = \dfrac{d^{1}s}{d^{0}s} - 1 = \sqrt{1 + 2({}^{1}L)} - 1$

Dynamic strain $\quad \bar{\varepsilon} = \dfrac{d^{2}s - d^{1}s}{d^{0}s} = \sqrt{1 + 2({}^{2}L)} - \sqrt{1 + 2({}^{1}L)} \quad$ (2 a-c)

The Green strains in each state that represents in equation (2) can be derived in the terms of displacements of the riser as follows.

$$ {}^{2}L = \frac{1}{({}^{0}s')^2}\left({}^{0}x'({}^{2}u') + {}^{0}y'({}^{2}v') + {}^{0}z'({}^{2}w') + \frac{({}^{2}u')^2}{2} + \frac{({}^{2}v')^2}{2} + \frac{({}^{2}w')^2}{2} \right) $$

$$ {}^{1}L = \frac{1}{({}^{0}s')^2}\left({}^{0}x'({}^{1}u') + {}^{0}y'({}^{1}v') + {}^{0}z'({}^{1}w') + \frac{({}^{1}u')^2}{2} + \frac{({}^{1}v')^2}{2} + \frac{({}^{1}w')^2}{2} \right) $$

$$ L = {}^{2}L - {}^{1}L = \frac{1}{{}^{0}s'^2}\left({}^{1}x'u' + {}^{1}y'v' + {}^{1}z'w' + \frac{u'^2}{2} + \frac{v'^2}{2} + \frac{w'^2}{2} \right) \quad \text{(3 a-c)} $$

Updated Lagrangian Descriptor (ULD)

The coordinate that follows motion and deformation of a deformable body with respect to position, direction, and size of the body at the intermediate state (or equilibrium state, the last known deformed configuration herein) is said to be the updated Lagrangian descriptor.

Total strain $\quad {}^{2}\varepsilon = \dfrac{d^{2}s - d^{0}s}{d^{1}s} = \sqrt{1 + 2\upsilon} - \sqrt{1 - 2({}^{1}\upsilon)}$

Static strain $\quad {}^{1}\varepsilon = \dfrac{d^{1}s - d^{0}s}{d^{1}s} = 1 - \dfrac{d^{0}s}{d^{1}s} = 1 - \sqrt{1 - 2({}^{1}\upsilon)}$

Dynamic strain $\quad \varepsilon = \dfrac{d^{2}s - d^{1}s}{d^{1}s} = \dfrac{d^{2}s}{d^{1}s} - 1 = \sqrt{1 + 2\upsilon} - 1 \quad$ (4 a-c)

The updated Green strains in each state that represents in equation (4) can be derived in the term of displacements of the riser which relate to the Green strains as

$$ {}^{2}\upsilon = {}^{1}\upsilon + \upsilon = {}^{2}L\left(\frac{{}^{0}s'}{{}^{1}s'}\right)^2, \quad {}^{1}\upsilon = {}^{1}L\left(\frac{{}^{0}s'}{{}^{1}s'}\right)^2, \quad \upsilon = {}^{2}\upsilon - {}^{1}\upsilon = L\left(\frac{{}^{0}s'}{{}^{1}s'}\right)^2 \quad \text{(5 a-c)} $$

Eulerian Descriptor (ED)

The coordinate that follows motion and deformation of a deformable body with respect to position, direction, and size of the body at the final state (or the displaced state herein) is said to be the Eulerian descriptor (ED).

Total strain $${}^{2E}\varepsilon = \frac{d\,{}^{2}s - d\,{}^{0}s}{d\,{}^{2}s} = 1 - \frac{d\,{}^{0}s}{d\,{}^{2}s} = 1 - \sqrt{1 - 2\left({}^{2}E\right)} \quad (6\text{ a})$$

Static strain $${}^{1E}\varepsilon = \frac{d\,{}^{1}s - d\,{}^{0}s}{d\,{}^{2}s} = \sqrt{1 - 2E} - \sqrt{1 - 2\left({}^{2}E\right)} \quad (6\text{ b})$$

Dynamic strain $${}^{E}\varepsilon = \frac{d\,{}^{2}s - d\,{}^{1}s}{d\,{}^{2}s} = 1 - \frac{d\,{}^{1}s}{d\,{}^{2}s} = 1 - \sqrt{1 - 2E} \quad (6\text{ c})$$

The Almansi strains in each state that represents in equation (6) can be derived in terms of displacements of the riser which relate to the Green strains as

$${}^{2}E = {}^{2}L\left(\frac{{}^{0}s'}{{}^{2}s'}\right)^{2},\ {}^{1}E = {}^{1}L\left(\frac{{}^{0}s'}{{}^{2}s'}\right)^{2},\ E = {}^{2}E - {}^{1}E = L\left(\frac{{}^{0}s'}{{}^{2}s'}\right)^{2} \quad (7\text{ a-c})$$

PROPERTIES OF THE PIPE AND TRANSPORTING FLUID IN THREE DEFORMATION DESCRIPTORS

The change of the large axial strain among three states leads to relations of differential arc-length of the pipe, cross-sectional properties of the pipe and internal flow velocity of transported fluid is shown as follows.

a) *Relations of differential arc-length of the pipe*

TLD; $$d\,{}^{0}s = \frac{d\,{}^{1}s}{1 + {}^{1}\bar{\varepsilon}} = \frac{d\,{}^{2}s}{1 + {}^{2}\bar{\varepsilon}} \quad (8\text{ a})$$

ULD; $$\frac{d\,{}^{0}s}{1 - {}^{1}\varepsilon} = d\,{}^{1}s = \frac{d\,{}^{2}s}{1 + \varepsilon} \quad (8\text{ b})$$

ED: $$\frac{d\,{}^{0}s}{1 - {}^{2E}\varepsilon} = \frac{d\,{}^{1}s}{1 - {}^{E}\varepsilon} = d\,{}^{2}s \quad (8\text{ c})$$

b) *Relations of cross-sectional properties of the pipe*

Since the pipe volume is conserved, the cross-sectional areas of the pipe at the three states, ${}^{i}A_p$, can be related to each other as

TLD; $${}^{0}A_p = {}^{1}A_p\left(1 + {}^{1}\bar{\varepsilon}\right) = {}^{2}A_p\left(1 + {}^{2}\bar{\varepsilon}\right) \quad (9\text{ a})$$

ULD; $${}^{0}A_p = \frac{{}^{1}A_p}{\left(1 - {}^{1}\varepsilon\right)} = \frac{{}^{2}A_p\left(1 + \varepsilon\right)}{\left(1 - {}^{1}\varepsilon\right)} \quad (9\text{ b})$$

ED; $${}^{0}A_p = \frac{{}^{1}A_p}{\left(1 - {}^{1E}\varepsilon\right)} = \frac{{}^{2}A_p}{\left(1 - {}^{2E}\varepsilon\right)} \quad (9\text{ c})$$

The relations of diameter, $\left({}^{i}D_p\right)$, moment of inertia, $\left({}^{i}I_p\right)$, and polar moment of inertia, $\left({}^{i}J_p\right)$, of the circular pipe among the three states determined corresponding to equations (9 a-c) are shown below.

TLD; $${}^{0}D_p = {}^{1}D_p\sqrt{1 + {}^{1}\bar{\varepsilon}} = {}^{2}D_p\sqrt{1 + {}^{2}\bar{\varepsilon}} \quad (10\text{ a})$$

$${}^{0}I_p = {}^{1}I_p\left(1 + {}^{1}\bar{\varepsilon}\right)^{2} = {}^{2}I_p\left(1 + {}^{2}\bar{\varepsilon}\right)^{2} \quad (10\text{ b})$$

$${}^{0}J_p = {}^{1}J_p\left(1 + {}^{1}\bar{\varepsilon}\right)^{2} = {}^{2}J_p\left(1 + {}^{2}\bar{\varepsilon}\right)^{2} \quad (10\text{ c})$$

ULD; $${}^{0}D_p = \frac{{}^{1}D_p}{\sqrt{1 - {}^{1}\varepsilon}} = {}^{2}D_p\sqrt{\frac{1 + \varepsilon}{1 - {}^{1}\varepsilon}} \quad (11\text{ a})$$

$${}^{0}I_p = \frac{{}^{1}I_p}{\left(1 - {}^{1}\varepsilon\right)^{2}} = {}^{2}I_p\frac{\left(1 + \varepsilon\right)^{2}}{\left(1 - {}^{1}\varepsilon\right)^{2}} \quad (11\text{ b})$$

$${}^{0}J_p = \frac{{}^{1}J_p}{\left(1 - {}^{1}\varepsilon\right)^{2}} = {}^{2}J_p\frac{\left(1 + \varepsilon\right)^{2}}{\left(1 - {}^{1}\varepsilon\right)^{2}} \quad (11\text{ c})$$

ED; $${}^{0}D_p = \frac{{}^{1}D_p}{\sqrt{1 - {}^{1E}\varepsilon}} = \frac{{}^{2}D_p}{\sqrt{1 - {}^{2E}\varepsilon}} \quad (12\text{ a})$$

$${}^{0}I_p = \frac{{}^{1}I_p}{\left(1 - {}^{1E}\varepsilon\right)^{2}} = \frac{{}^{2}I_p}{\left(1 - {}^{2E}\varepsilon\right)^{2}} \quad (12\text{ b})$$

$${}^{0}J_p = \frac{{}^{1}J_p}{\left(1 - {}^{1E}\varepsilon\right)^{2}} = \frac{{}^{2}J_p}{\left(1 - {}^{2E}\varepsilon\right)^{2}} \quad (12\text{ c})$$

c) *Relations of internal flow velocity of transported fluid*

By substituting equation (9) into the continuity equation of the fluid flow in the control volume pipe, the relationships of internal flow velocities at the three states are obtained as

TLD; $${}^{0}V_i = \frac{{}^{1}V_i}{1 + {}^{1}\bar{\varepsilon}} = \frac{{}^{2}V_i}{1 + {}^{2}\bar{\varepsilon}} \quad (13\text{ a})$$

ULD; $${}^{0}V_i = {}^{1}V_i\left(1 - {}^{1}\varepsilon\right) = \frac{{}^{2}V_i\left(1 - {}^{1}\varepsilon\right)}{\left(1 + \varepsilon\right)} \quad (13\text{ b})$$

ED: $${}^{0}V_i = {}^{1}V_i\left(1 - {}^{1E}\varepsilon\right) = {}^{2}V_i\left(1 - {}^{2E}\varepsilon\right) \quad (13\text{ c})$$

THE EXTENSIBLE ELASTICA THEORY

The followings are the extensible elastica theorems for the Hookean material pipe corresponding to the three deformation descriptors (Chucheepsakul et al., 2003). These theorems are used to develop the large strain formulations of three-dimensional extensible flexible pipe, which will be discussed later.

Theorem 1: When the TLD is adopted to describe deformation of the pipe, the fiber strain, the constitutive relations and the virtual strain energy are expressed as follows

$${}^{2}\bar{\varepsilon}_{\zeta} = {}^{2}\bar{\varepsilon} + \zeta\left[{}^{2}\kappa\left(1 + {}^{2}\bar{\varepsilon}\right) - {}^{0}\kappa\right]$$

$${}^{2}N = E\,{}^{0}A_p\,{}^{2}\bar{\varepsilon},\ {}^{2}M = E\,{}^{0}I_p\left[{}^{2}\kappa\left(1 + {}^{2}\bar{\varepsilon}\right) - {}^{0}\kappa\right],$$

$${}^{2}T = G\,{}^{0}J_p\left[{}^{2}\tau\left(1 + {}^{2}\bar{\varepsilon}\right) - {}^{0}\tau\right],$$

$$\delta U = \int_{{}^{0}s}\left\{{}^{2}N\delta\,{}^{2}\bar{\varepsilon} + {}^{2}M\delta\left[{}^{2}\kappa\left(1 + {}^{2}\bar{\varepsilon}\right) - {}^{0}\kappa\right] + {}^{2}T\delta\left[{}^{2}\tau\left(1 + {}^{2}\bar{\varepsilon}\right) - {}^{0}\tau\right]\right\}d\,{}^{2}s$$

$$\delta U = \int_{\alpha}\left[{}^{2}N\delta\,{}^{2}s' + {}^{2}M\delta\left({}^{2}\theta' - {}^{0}\theta'\right) + {}^{2}T\delta\left({}^{2}\phi' - {}^{0}\phi'\right)\right]d\alpha$$

(14 a-f)

Theorem 2: When the ULD is adopted to describe deformation of the pipe, the fiber strain, the constitutive relations and the virtual strain energy are expressed as follows

$$^{I}\varepsilon_{\zeta} = {}^{I}\varepsilon + \zeta\left[{}^{I}\kappa(I+\varepsilon) - {}^{o}\kappa(I-{}^{I}\varepsilon)\right]$$
$${}^{I}N = E{}^{I}A_{p}{}^{I}\varepsilon\,,\ {}^{I}M = E{}^{I}I_{p}\left[{}^{I}\kappa(I+\varepsilon) - {}^{o}\kappa(I-{}^{I}\varepsilon)\right],$$
$${}^{I}T = G{}^{I}J_{p}\left[{}^{I}\tau(I+\varepsilon) - {}^{o}\tau(I-{}^{I}\varepsilon)\right],$$
$$\delta U = \int_{{}^{I}s}\Big\{{}^{I}N\delta{}^{I}\varepsilon + {}^{I}M\delta\left[{}^{I}\kappa(I+\varepsilon) - {}^{o}\kappa(I-{}^{I}\varepsilon)\right]$$
$$+{}^{I}T\delta\left[{}^{I}\tau(I+\varepsilon) - {}^{o}\tau(I-{}^{I}\varepsilon)\right]\Big\}d{}^{I}s$$
$$\delta U = \int_{\alpha}\left[{}^{I}N\delta{}^{I}s' + {}^{I}M\delta({}^{I}\theta' - {}^{o}\theta') + {}^{I}T\delta({}^{I}\phi' - {}^{o}\phi')\right]d\alpha$$
(15 a-f)

Theorem 3: When the ED is adopted to describe deformation of the pipe, the fiber strain, the constitutive relations and the virtual strain energy are expressed as follows

$${}^{IE}\varepsilon_{\zeta} = {}^{IE}\varepsilon + \zeta\left[{}^{I}\kappa - {}^{o}\kappa(I-{}^{IE}\varepsilon)\right]$$
$${}^{I}N = E{}^{I}A_{p}{}^{IE}\varepsilon\,,\ {}^{I}M = E{}^{I}I_{p}\left[{}^{I}\kappa - {}^{o}\kappa(I-{}^{IE}\varepsilon)\right],$$
$${}^{I}T = G{}^{I}J_{p}\left[{}^{I}\tau - {}^{o}\tau(I-{}^{IE}\varepsilon)\right],$$
$$\delta U = \int_{{}^{I}s}\Big\{{}^{I}N\delta{}^{I}\overline{\varepsilon} + {}^{I}M\delta\left[{}^{I}\kappa - {}^{o}\kappa(I-{}^{IE}\varepsilon)\right]$$
$$+{}^{I}T\delta\left[{}^{I}\tau - {}^{o}\tau(I-{}^{IE}\varepsilon)\right]\Big\}d{}^{I}s$$
$$\delta U = \int_{\alpha}\left[{}^{I}N\delta{}^{I}s' + {}^{I}M\delta({}^{I}\theta' - {}^{o}\theta') + {}^{I}T\delta({}^{I}\phi' - {}^{o}\phi')\right]d\alpha$$
(16 a-f)

in which ε_{ζ} is the axial strain at any fiber radius (ζ), E is the elastic modulus, G is the shear modulus, N is the axial force, M is the bending moment, T is the torque, and U is the strain energies due to axial force, bending moment, and torsion of the pipe.

THE APPARENT TENSION AND THE APPARENT WEIGHT

The effects of tension, pressure and weight on pipe behavior have been studied for more than a century. There are many ways to derive these effects which treated in numerous textbooks. According to Chucheepsakul et al. (2003), the apparent tension and the apparent weight are derived from the first law of Archimedes. Since the Archimedes' principle is usable with the enclosing pressure fields, the technique of superimposition is adopted to determine tension and weight of the pipe on the real system.

The expressions for the apparent weight and the apparent tension generally for the three deformation descriptors are

$$w_{a} = \left(\rho_{p}{}^{I}A_{p} - \rho_{e}{}^{I}A_{e} + \rho_{i}{}^{I}A_{i}\right)g \tag{17}$$
$$N_{a} = E{}^{I}A_{p}{}^{I}\varepsilon = N + 2v\left(p_{e}{}^{I}A_{e} - p_{i}{}^{I}A_{i}\right) \tag{18}$$

in which N is the true-wall tension or tension of an empty pipe in the air, ρ_p, ρ_e, and ρ_i are densities of a pipe, external fluid, and internal fluid respectively, v is the Poisson's ratio, and g is the gravitational acceleration.

DYNAMIC INTERACTIONS BETWEEN FLUID AND PIPE

In this section, influence of dynamic pressures due to flow of internal and external fluids is considered.

Hydrodynamic forces due to cross-flows of current and waves.

The hydrodynamic forces exerted on flexible marine risers with large displacements in the orthogonal triad system based on the coupled Morison equation (Chakrabarti, 1990) can be expressed as

$$\mathrm{F}_{H} = \begin{Bmatrix} f_{Ht} \\ f_{Hn} \\ f_{Hbn} \end{Bmatrix} = 0.5\rho_{e}{}^{I}D_{e}\begin{Bmatrix} \pi C_{Dt}\gamma_{t}|\gamma_{t}| \\ C_{Dn}\gamma_{n}|\gamma_{n}| \\ C_{Dbn}\gamma_{bn}|\gamma_{bn}| \end{Bmatrix} + \rho_{e}{}^{I}A_{e}C_{a}\begin{Bmatrix} \dot{\gamma}_{t} \\ \dot{\gamma}_{n} \\ \dot{\gamma}_{bn} \end{Bmatrix} + \rho_{e}{}^{I}A_{e}\begin{Bmatrix} \dot{V}_{Ht} \\ \dot{V}_{Hn} \\ \dot{V}_{Hbn} \end{Bmatrix} \tag{19}$$

where C_{Dt}, C_{Dn}, and C_{Dbn} are the tangential, normal, and binormal drag coefficients; C_a is the added mass coefficient; V_{Ht}, V_{Hn}, and V_{Hbn} are the tangential, normal and binormal velocities of currents and waves; and $\gamma_t = V_{Ht} - \dot{u}_t$, $\gamma_n = V_{Hn} - \dot{v}_n$, and $\gamma_{bn} = V_{Hbn} - \dot{w}_{bn}$ are the velocities of currents and waves relative to pipe velocities $\dot{u}_t, \dot{v}_n$, and $\dot{w}_{bn}$ in tangential, normal, and binormal directions, respectively. For large strain analysis, the effect of cross-sectional changes of the pipe in equation (9) has to be applied to equation (19).

To eliminate the difficulty of operating with absolute function in equation (19), the signum function is used. Here

$$sgn(\gamma) = \begin{cases} I & if\,\gamma \geq 0 \\ -I & if\,\gamma < 0. \end{cases} \tag{20}$$

With some manipulations, equation (19) can be arranged into

$$\mathrm{F}_{H} = \begin{Bmatrix} f_{Ht} \\ f_{Hn} \\ f_{Hbn} \end{Bmatrix} = -\begin{bmatrix} C_{a}^{*} & 0 & 0 \\ 0 & C_{a}^{*} & 0 \\ 0 & 0 & C_{a}^{*} \end{bmatrix}\begin{Bmatrix} \ddot{u}_{t} \\ \ddot{v}_{n} \\ \ddot{w}_{bn} \end{Bmatrix} - \begin{bmatrix} C_{eqt}^{*} & 0 & 0 \\ 0 & C_{eqn}^{*} & 0 \\ 0 & 0 & C_{eqbn}^{*} \end{bmatrix}\begin{Bmatrix} \dot{u}_{t} \\ \dot{v}_{n} \\ \dot{w}_{bn} \end{Bmatrix} + \begin{Bmatrix} C_{Dt}^{*}V_{Ht}^{2} + C_{M}^{*}\dot{V}_{Ht} \\ C_{Dn}^{*}V_{Hn}^{2} + C_{M}^{*}\dot{V}_{Hn} \\ C_{Dbn}^{*}V_{Hbn}^{2} + C_{M}^{*}\dot{V}_{Hbn} \end{Bmatrix} \tag{21}$$

where the coefficients of equivalent tangential damping C_{eqt}^{*}, tangential drag forces C_{Dt}^{*}, equivalent normal damping C_{eqn}^{*}, normal drag forces C_{Dn}^{*}, equivalent binormal damping C_{eqbn}^{*}, binormal drag forces C_{Dbn}^{*}, and the equivalent coefficients of added mass C_{a}^{*} and inertia forces C_{M}^{*} are

$$C_{eqt}^{*} = C_{Dt}^{*}\left[2V_{Ht} - \dot{u}_{t}\right],\ C_{Dt}^{*} = 0.5\rho_{e}{}^{I}D_{e}\pi C_{Dt}\cdot sgn(\gamma_{t}) \quad \text{(22 a,b)}$$
$$C_{eqn}^{*} = C_{Dn}^{*}\left[2V_{Hn} - \dot{v}_{n}\right],\ C_{Dn}^{*} = 0.5\rho_{e}{}^{I}D_{e}C_{Dn}\cdot sgn(\gamma_{n}) \quad \text{(22 c,d)}$$
$$C_{eqbn}^{*} = C_{Dbn}^{*}\left[2V_{Hbn} - \dot{w}_{bn}\right],\ C_{Dbn}^{*} = 0.5\rho_{e}{}^{I}D_{e}C_{Dbn}\cdot sgn(\gamma_{bn}) \quad \text{(22 f,g)}$$
$$C_{a}^{*} = \rho_{e}{}^{I}A_{e}C_{a}\,,\ C_{M}^{*} = \rho_{e}{}^{I}A_{e}C_{M} \quad \text{(22 h,i)}$$

in which C_a is the added mass coefficient and $C_M = I + C_a$ is the inertia coefficient.

In order to transform hydrodynamic force in the orthogonal triad system to the fixed Cartesian coordinate system, Euler's angle (Atanackovic, 1997) is used to find the transformation matrix, which is the orthogonal matrix and can be written as

$$\begin{Bmatrix} t \\ n \\ b \end{Bmatrix} = \begin{bmatrix} a_{1X} & a_{1Y} & a_{1Z} \\ a_{2X} & a_{2Y} & a_{2Z} \\ a_{3X} & a_{3Y} & a_{3Z} \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{23}$$

where

$$a_{1X} = \cos{}^2\vartheta_2 \cos{}^2\vartheta_3 \tag{24 a}$$

$$a_{1Y} = \cos{}^2\vartheta_2 \sin{}^2\vartheta_3 \cos{}^2\vartheta_1 + \sin{}^2\vartheta_2 \sin{}^2\vartheta_1 \tag{24 b}$$

$$a_{1Z} = \cos{}^2\vartheta_2 \sin{}^2\vartheta_3 \sin{}^2\vartheta_1 - \sin{}^2\vartheta_2 \cos{}^2\vartheta_1 \tag{24 c}$$

$$a_{2X} = -\sin{}^2\vartheta_3 \tag{24 d}$$

$$a_{2Y} = \cos{}^2\vartheta_3 \cos{}^2\vartheta_1 \tag{24 e}$$

$$a_{2Z} = \sin{}^2\vartheta_1 \cos{}^2\vartheta_3 \tag{24 f}$$

$$a_{3X} = \cos{}^2\vartheta_3 \sin{}^2\vartheta_2 \tag{24 g}$$

$$a_{3Y} = \sin{}^2\vartheta_2 \sin{}^2\vartheta_3 \cos{}^2\vartheta_1 - \cos{}^2\vartheta_2 \sin{}^2\vartheta_1 \tag{24 h}$$

$$a_{3Z} = \sin{}^2\vartheta_1 \sin{}^2\vartheta_2 \cos{}^2\vartheta_3 + \cos{}^2\vartheta_1 \cos{}^2\vartheta_2 \tag{24 i}$$

Thus, equation (21) can be transformed into the fixed Cartesian coordinates system as

$$\bar{F}_H = \begin{Bmatrix} f_{Hx} \\ f_{Hy} \\ f_{Hz} \end{Bmatrix} = -\begin{bmatrix} C^*_a & 0 & 0 \\ 0 & C^*_a & 0 \\ 0 & 0 & C^*_a \end{bmatrix} \begin{Bmatrix} {}^2\ddot{x} \\ {}^2\ddot{y} \\ {}^2\ddot{z} \end{Bmatrix} - \begin{bmatrix} C^*_{eqx} & C^*_{eqxy} & C^*_{eqxz} \\ C^*_{eqxy} & C^*_{eqy} & C^*_{eqyz} \\ C^*_{eqxz} & C^*_{eqyz} & C^*_{eqz} \end{bmatrix} \begin{Bmatrix} {}^2\dot{x} \\ {}^2\dot{y} \\ {}^2\dot{z} \end{Bmatrix}$$

$$+\begin{Bmatrix} C^*_m \dot{V}_{Hx} + C^*_{Dx} V^2_{Hx} + 2C^*_{Dxy1} V_{Hx}V_{Hy} + 2C^*_{Dxz1} V_{Hx}V_{Hz} + 2C^*_{Dxyz} V_{Hy}V_{Hz} + C^*_{Dxy2} V^2_{Hy} + C^*_{Dxz2} V^2_{Hz} \\ C^*_m \dot{V}_{Hy} + C^*_{Dy} V^2_{Hy} + 2C^*_{Dxy1} V_{Hx}V_{Hy} + 2C^*_{Dxyz} V_{Hx}V_{Hz} + 2C^*_{Dyz1} V_{Hy}V_{Hz} + C^*_{Dxy1} V^2_{Hx} + C^*_{Dyz2} V^2_{Hz} \\ C^*_m \dot{V}_{Hz} + C^*_{Dz} V^2_{Hz} + +2C^*_{Dxyz} V_{Hx}V_{Hy} + 2C^*_{Dxz2} V_{Hx}V_{Hz} + 2C^*_{Dyz2} V_{Hy}V_{Hz} + C^*_{Dxz1} V^2_{Hx} + C^*_{Dyz1} V^2_{Hy} \end{Bmatrix} \tag{25}$$

where V_{Hx}, V_{Hy} and V_{Hz} are the velocities of external fluid in x, y, and z directions respectively, and

$$\left.\begin{aligned} C^*_{eqx} &= C^*_{eqt}a^2_{1X} + C^*_{eqn}a^2_{2X} + C^*_{eqbn}a^2_{3X} \\ C^*_{eqy} &= C^*_{eqt}a^2_{1Y} + C^*_{eqn}a^2_{2Y} + C^*_{eqbn}a^2_{3Y} \\ C^*_{eqz} &= C^*_{eqt}a^2_{1Z} + C^*_{eqn}a^2_{2Z} + C^*_{eqbn}a^2_{3Z} \end{aligned}\right\} \tag{26 a-c}$$

$$\left.\begin{aligned} C^*_{eqxy} &= C^*_{eqt}a_{1X}a_{1Y} + C^*_{eqn}a_{2X}a_{2Y} + C^*_{eqbn}a_{3X}a_{3Y} \\ C^*_{eqxz} &= C^*_{eqt}a_{1X}a_{1Z} + C^*_{eqn}a_{2X}a_{2Z} + C^*_{eqbn}a_{3X}a_{3Z} \\ C^*_{eqyz} &= C^*_{eqt}a_{1Y}a_{1Z} + C^*_{eqn}a_{2Y}a_{2Z} + C^*_{eqbn}a_{3Y}a_{3Z} \end{aligned}\right\} \tag{27 a-c}$$

$$\left.\begin{aligned} C^*_{Dx} &= C^*_{Dt}a^3_{1X} + C^*_{Dn}a^3_{2X} + C^*_{Dbn}a^3_{3X} \\ C^*_{Dy} &= C^*_{Dt}a^3_{1Y} + C^*_{Dn}a^3_{2Y} + C^*_{Dbn}a^3_{3Y} \\ C^*_{Dz} &= C^*_{Dt}a^3_{1Z} + C^*_{Dn}a^3_{2Z} + C^*_{Dbn}a^3_{3Z} \end{aligned}\right\} \tag{28 a-c}$$

$$\left.\begin{aligned} C^*_{Dxy1} &= C^*_{Dt}a^2_{1X}a_{1Y} + C^*_{Dn}a^2_{2X}a_{2Y} + C^*_{Dbn}a^2_{3X}a_{3Y} \\ C^*_{Dxz1} &= C^*_{Dt}a^2_{1X}a_{1Z} + C^*_{Dn}a^2_{2X}a_{2Z} + C^*_{Dbn}a^2_{3X}a_{3Z} \\ C^*_{Dyz1} &= C^*_{Dt}a^2_{1Y}a_{1Z} + C^*_{Dn}a^2_{2Y}a_{2Z} + C^*_{Dbn}a^2_{3Y}a_{3Z} \\ C^*_{Dxy2} &= C^*_{Dt}a_{1X}a^2_{1Y} + C^*_{Dn}a_{2X}a^2_{2Y} + C^*_{Dbn}a_{3X}a^2_{3Y} \\ C^*_{Dxz2} &= C^*_{Dt}a_{1X}a^2_{1Z} + C^*_{Dn}a_{2X}a^2_{2Z} + C^*_{Dbn}a_{3X}a^2_{3Z} \\ C^*_{Dyz2} &= C^*_{Dt}a_{1Y}a^2_{1Z} + C^*_{Dn}a_{2Y}a^2_{2Z} + C^*_{Dbn}a_{3Y}a^2_{3Z} \\ C^*_{Dxyz} &= C^*_{Dt}a_{1X}a_{1Y}a_{1Z} + C^*_{Dn}a_{2X}a_{2Y}a_{2Z} + C^*_{Dbn}a_{3X}a_{3Y}a_{3Z} \end{aligned}\right\} \tag{29 a-g}$$

Equations (26 a-c) represent the coefficients of equivalent hydrodynamic damping force in x, y, and z directions. Equations (27 a-c) represent the coefficients of equivalent hydrodynamic damping force in x-y, x-z, and y-z planes. Equations (28 a-c) represent the coefficients of drag force in x, y, and z directions. Equations (29 a-g) represent the coefficients of drag force in x-y, x-z, and y-z planes.

At the equilibrium state, static loading is due only to the steady flow of external fluid. Therefore, the hydrodynamic forces from equations (21) and (25) are reduced to

$${}^1\bar{F}_H = \begin{Bmatrix} {}^1f_{Ht} \\ {}^1f_{Hn} \\ {}^1f_{Hbn} \end{Bmatrix} = \begin{Bmatrix} {}^1C^*_{Dt}\,{}^1V^2_{mt} \\ {}^1C^*_{Dn}\,{}^1V^2_{mn} \\ {}^1C^*_{Dbn}\,{}^1V^2_{mbn} \end{Bmatrix} \tag{30}$$

$${}^1\bar{F}_H = \begin{Bmatrix} {}^1f_{Hx} \\ {}^1f_{Hy} \\ {}^1f_{Hz} \end{Bmatrix} = \begin{Bmatrix} {}^1C^*_{Dx}\,{}^1V^2_{Hx} + 2\,{}^1C^*_{Dxy1}\,{}^1V_{Hx}\,{}^1V_{Hy} + 2\,{}^1C^*_{Dxz1}\,{}^1V_{Hx}\,{}^1V_{Hz} + 2\,{}^1C^*_{Dxyz}\,{}^1V_{Hy}\,{}^1V_{Hz} + {}^1C^*_{Dxy2}\,{}^1V^2_{Hy} + {}^1C^*_{Dxz2}\,{}^1V^2_{Hz} \\ {}^1C^*_{Dy}\,{}^1V^2_{Hy} + 2\,{}^1C^*_{Dxy1}\,{}^1V_{Hx}\,{}^1V_{Hy} + 2\,{}^1C^*_{Dxyz}\,{}^1V_{Hx}\,{}^1V_{Hz} + 2\,{}^1C^*_{Dyz1}\,{}^1V_{Hy}\,{}^1V_{Hz} + {}^1C^*_{Dxy1}\,{}^1V^2_{Hx} + {}^1C^*_{Dyz2}\,{}^1V^2_{Hz} \\ {}^1C^*_{Dz}\,{}^1V^2_{Hz} + +2\,{}^1C^*_{Dxyz}\,{}^1V_{Hx}\,{}^1V_{Hy} + 2\,{}^1C^*_{Dxz2}\,{}^1V_{Hx}\,{}^1V_{Hz} + 2\,{}^1C^*_{Dyz2}\,{}^1V_{Hy}\,{}^1V_{Hz} + {}^1C^*_{Dxz1}\,{}^1V^2_{Hx} + {}^1C^*_{Dyz1}\,{}^1V^2_{Hy} \end{Bmatrix} \tag{31}$$

Hydrodynamic forces due to internal flow of transported fluid

The velocity and acceleration of transported fluid can be derived as (Huang, 1993),

$$\bar{V}_F = \bar{V}_P + \bar{V}_{FP} = \frac{\partial \bar{r}_P}{\partial t} + \frac{V_{FP}}{{}^2s'}\frac{\partial \bar{r}_P}{\partial \alpha} \tag{32}$$

$$\bar{a}_F = \bar{a}_P + \bar{a}_{FP} = \frac{D\bar{V}_P}{Dt} + \frac{D\bar{V}_{FP}}{Dt} = \frac{D}{Dt}\left(\frac{\partial \bar{r}_P}{\partial t}\right) + \frac{D}{Dt}\left(\frac{V_{FP}}{{}^2s'}\frac{\partial \bar{r}_P}{\partial \alpha}\right)$$

$$= \underbrace{\frac{\partial^2 \bar{r}_P}{\partial t^2}}_{(1)} + \underbrace{\left(\frac{2V_{FP}}{{}^2s'}\right)\frac{\partial^2 \bar{r}_P}{\partial \alpha \partial t}}_{(2)} + \underbrace{\left(\frac{V_{FP}}{{}^2s'}\right)^2 \frac{\partial^2 \bar{r}_P}{\partial \alpha^2}}_{(3)} + \left[\underbrace{\frac{\dot{V}_{FP}}{{}^2s'}}_{(4)} + \underbrace{\frac{V_{FP}V'_{FP}}{({}^2s')^2}}_{(5)} \underbrace{- \frac{V_{FP}\,{}^2\dot{s}'}{({}^2s')^2} - \frac{V^2_{FP}\,{}^2s''}{({}^2s')^3}}_{(6)}\right]\frac{\partial \bar{r}_P}{\partial \alpha} \tag{33}$$

in which the term (1) is the transported mass acceleration, (2) is the coriolis acceleration, (3) is the centripetal acceleration, (4) is the local acceleration due to unsteady flow, (5) is the convective acceleration due to non-uniform flow, and (6) is the relative accelerations due to local coordinate rotation and displacement.

By using the differential geometry formulas given in appendix and let V_t be the relative velocity of the transporting fluid, i.e. $V_t = V_{FP}$, the velocity and acceleration of transported fluid in the fixed Cartesian coordinate system can be expressed as follows

$$\bar{V}_F = \left[{}^2\dot{x} + \frac{V_t\,{}^2x'}{{}^2s'}\right]\hat{i} + \left[{}^2\dot{y} + \frac{V_t\,{}^2y'}{{}^2s'}\right]\hat{j} + \left[{}^2\dot{z} + \frac{V_t\,{}^2z'}{{}^2s'}\right]\hat{k} \tag{34}$$

$$\bar{a}_F = \left\{ {}^2\ddot{x} + \left[\left(\frac{2}{{}^2s'} - \frac{({}^2x')^2}{({}^2s')^3}\right){}^2\dot{x}' - \left(\frac{{}^2x'\,{}^2y'}{({}^2s')^3}\right){}^2\dot{y}' - \left(\frac{{}^2x'\,{}^2z'}{({}^2s')^3}\right){}^2\dot{z}'\right]V_t \right.$$

$$\left. + \left[\frac{({}^2x''\,{}^2y' - {}^2x'\,{}^2y'')\,{}^2y' + ({}^2x''\,{}^2z' - {}^2x'\,{}^2z'')\,{}^2z'}{({}^2s')^4}\right]V_t^2 + \left(\frac{DV_t}{Dt}\right)\frac{{}^2x'}{{}^2s'} \right\}\hat{i}$$

$$+ \left\{ {}^2\ddot{y} + \left[-\left(\frac{{}^2x'\,{}^2y'}{({}^2s')^3}\right){}^2\dot{x}' + \left(\frac{2}{{}^2s'} - \frac{({}^2y')^2}{({}^2s')^3}\right){}^2\dot{y}' - \left(\frac{{}^2y'\,{}^2z'}{({}^2s')^3}\right){}^2\dot{z}'\right]V_t \right.$$

$$\left. + \left[\frac{({}^2y''\,{}^2x' - {}^2y'\,{}^2x'')\,{}^2x' + ({}^2y''\,{}^2z' - {}^2y'\,{}^2z''),{}^2z'}{({}^2s')^4}\right]V_t^2 + \left(\frac{DV_t}{Dt}\right)\frac{{}^2y'}{{}^2s'} \right\}\hat{j}$$

$$+ \left\{ {}^2\ddot{z} + \left[-\left(\frac{{}^2x'\,{}^2z'}{({}^2s')^3}\right){}^2\dot{x}' - \left(\frac{{}^2y'\,{}^2z'}{({}^2s')^3}\right){}^2\dot{y}' + \left(\frac{2}{{}^2s'} - \frac{({}^2z')^2}{({}^2s')^3}\right){}^2\dot{z}'\right]V_t \right.$$

$$+\left[\frac{\left({}^2z''{}^2x'-{}^2z'{}^2x''\right){}^2x'+\left({}^2z''{}^2y'-{}^2z'{}^2y''\right){}^2y'}{\left({}^2s'\right)^4}\right]V_t^2+\left(\frac{DV_t}{Dt}\right)\frac{{}^2z'}{{}^2s'}\Bigg\}\hat{k} \tag{35}$$

in which
$$\frac{D(\ )}{Dt}=\frac{\partial(\ )}{\partial t}+\frac{V_t}{{}^2s'}\frac{\partial(\ )}{\partial\alpha} \tag{36}$$

VIRTUAL WORK FORMULATIONS

Based on the elastica theory, the apparent tension concept and dynamic interactions between fluid and pipe, the internal virtual work and external virtual work can be obtained.

Internal virtual work of the effective system

For the overall apparent system, the pipe is subjected to the apparent tension N_e in place of the axial force of the real system. Therefore, applying equations (14-16) (the extensible elastica theory) to the apparent system yields the stiffness equation of the initially straight pipe:

$$\delta U=\int_\alpha\left[{}^2N_e\delta{}^2s'+{}^2M\delta{}^2\theta'+{}^2T\delta{}^2\phi'\right]d\alpha \tag{37}$$

where

$${}^2N_e=\begin{cases}E^oA_p{}^2\bar{\varepsilon} & \text{(TLD)}\\ E^tA_p{}^2\varepsilon & \text{(ULD)}\\ E^2A_p{}^{2E}\varepsilon & \text{(ED)}\end{cases},\ {}^2M={}^2B{}^2\kappa,\ {}^2B=\begin{cases}E^oI_p(1+{}^2\bar{\varepsilon}) & \text{(TLD)}\\ E^tI_p(1+\varepsilon) & \text{(ULD)}\\ E^2I_p & \text{(ED)}\end{cases}$$

$${}^2T={}^2C{}^2\tau,\ {}^2C=\begin{cases}G^oJ_p(1+{}^2\bar{\varepsilon}) & \text{(TLD)}\\ G^tJ_p(1+\varepsilon) & \text{(ULD)}\\ G^2J_p & \text{(ELD)}\end{cases} \tag{38 a-c}$$

External virtual work of the effective system

The external forces exert upon the marine pipes are the effective weight, hydrodynamic loading, and inertial forces which depend on deformation of the pipe. Therefore, an evaluation of these forces should be done with respect to the current configuration of the pipe. Then the variation of external virtual work evaluating from the free bodies at displaced state is

$$\delta W=\delta W_w+\delta W_H+\delta W_I \tag{39}$$

where δW_w, δW_H and δW_I are the virtual work of the apparent weight, hydrodynamic pressure, and inertial forces of the pipe and transported fluid respectively. In the Cartesian coordinates, these expressions are written as follows,

$$\delta W_w=-\int_\alpha w_e{}^2s'\delta{}^2v\,d\alpha \tag{40}$$

$$\delta W_H=-\int_\alpha\left[\left(f_{Hx}{}^2s'\right)\delta{}^2u+\left(f_{Hy}{}^2s'\right)\delta{}^2v+\left(f_{Hz}{}^2s'\right)\delta{}^2w\right]d\alpha \tag{41}$$

$$\delta W_I=-\int_\alpha\Big[\left(m_pa_{px}+m_fa_{Fx}\right){}^2s'\delta{}^2u+\left(m_pa_{py}+m_fa_{Fy}\right){}^2s'\delta{}^2v$$
$$+\left(m_pa_{pz}+m_fa_{Fz}\right){}^2s'\delta{}^2w\Big]d\alpha \tag{42}$$

in which $\bar{a}_p=a_{px}\hat{i}+a_{py}\hat{j}+a_{pz}\hat{k}=\ddot{\bar{r}}={}_2\ddot{u}\hat{i}+{}_2\ddot{v}\hat{j}+{}_2\ddot{w}\hat{k}$ and the expressions of hydrodynamic force, $\bar{F}_H=f_{Hx}\hat{i}+f_{Hy}\hat{j}+f_{Hz}\hat{k}$, and the accelerate of transporting fluid, $\bar{a}_F=a_{Fx}\hat{i}+a_{Fy}\hat{j}+a_{Fz}\hat{k}$, are given by equations (25) and (35) respectively. Substituting equations (40)-(42) into equation (39) yields

$$\delta W=\int_\alpha\left\{{}^2s'\left[f_{Hx}-m_pa_{px}-m_fa_{Fx}\right]\delta{}^2u\right\}d\alpha$$
$$+\int_\alpha\left\{{}^2s'\left[-w_e+f_{Hy}-m_pa_{py}-m_fa_{Fy}\right]\delta{}^2v\right\}d\alpha$$
$$+\int_\alpha\left\{{}^2s'\left[f_{Hz}-m_pa_{pz}-m_fa_{Fz}\right]\delta{}^2w\right\}d\alpha \tag{43}$$

Total virtual work

From the principle of virtual work, the total virtual work of the effective system is zero:

$$\delta\pi=\delta U-\delta W=0 \tag{44}$$

Substituting equations (37) and (43) into equation (44) and utilizing the differential geometry expressions in appendix yields the total virtual work expressed in the fixed Cartesian coordinate. Integrating by part three times, one obtains the Euler's equation and the natural boundary conditions as follows.

$$\delta\pi=\underbrace{\left[{}^2T\left(\frac{{}^2b_x}{\left({}^2s'\right)^2{}^2\kappa}\delta{}^2u''+\frac{{}^2b_y}{\left({}^2s'\right)^2{}^2\kappa}\delta{}^2v''+\frac{{}^2b_z}{\left({}^2s'\right)^2{}^2\kappa}\delta{}^2w''\right)+\mathrm{F}_{2x}\delta{}^2u'+\mathrm{F}_{2y}\delta{}^2v'+\mathrm{F}_{2z}\delta{}^2z'\right]}_{\text{Torque boundary condition}}\Bigg|_{\alpha_o}^{\alpha_l}$$
$$+\underbrace{\left[\frac{{}^2M}{{}^2s'}\left({}^2n_x\delta{}^2u'+{}^2n_y\delta{}^2v'+{}^2n_z\delta{}^2w'\right)\right]}_{\text{Moment boundary condition}}\Bigg|_{\alpha_o}^{\alpha_l}$$
$$+\underbrace{\left[{}^2R_x\delta{}^2u'+{}^2R_y\delta{}^2v'+{}^2R_z\delta{}^2w'\right]}_{\text{Force boundary condition}}\Bigg|_{\alpha_o}^{\alpha_l}$$
$$+\underbrace{\int_\alpha\left[\begin{matrix}\left\{-{}^2R_x'-{}^2q_x\right\}\delta{}^2u+\left\{-{}^2R_y'-{}^2q_y\right\}\delta{}^2v\\+\left\{-{}^2R_z'-{}^2q_z\right\}\delta{}^2w\end{matrix}\right]d\alpha}_{\text{Euler's equation}} \tag{45}$$

where

$$\mathrm{F}_{2x}=\frac{-{}^2T\left({}^2y'{}^2z''-{}^2z'{}^2y''\right){}^2s''}{\left({}^2s'\right)^6\left({}^2\kappa\right)^2} \tag{46 a}$$

$$\mathrm{F}_{2y}=\frac{-{}^2T\left({}^2z'{}^2x''-{}^2x'{}^2z''\right){}^2s''}{\left({}^2s'\right)^6\left({}^2\kappa\right)^2} \tag{46 b}$$

$$\mathrm{F}_{2z}=\frac{-{}^2T\left({}^2x'{}^2y''-{}^2y'{}^2x''\right){}^2s''}{\left({}^2s'\right)^6\left({}^2\kappa\right)^2} \tag{46 c}$$

$${}^2R_x=\left[\left({}^2N_e-{}^2B\left({}^2\kappa\right)^2\right)\left(\frac{{}^2x'}{{}^2s'}\right)-\frac{{}^2B{}^2s''}{\left({}^2s'\right)^3}\frac{\partial}{\partial\alpha}\left(\frac{{}^2x'}{{}^2s'}\right)\right]+{}^2T{}^2\kappa{}^2b_x$$
$$-\left[\frac{{}^2B}{\left({}^2s'\right)^2}\frac{\partial}{\partial\alpha}\left(\frac{{}^2x'}{{}^2s'}\right)\right]' \tag{47 a}$$

$$ {}^2R_y = \left[\left({}^2N_a - {}^2B({}^2\kappa)^2\right)\left(\frac{{}^2y'}{{}^2s'}\right) - \frac{{}^2B\,{}^2s''}{({}^2s')^3}\frac{\partial}{\partial\alpha}\left(\frac{{}^2y'}{{}^2s'}\right)\right] + {}^2T\,{}^2\kappa\,{}^2b_y - \left[\frac{{}^2B}{({}^2s')^2}\frac{\partial}{\partial\alpha}\left(\frac{{}^2y'}{{}^2s'}\right)\right]' \qquad (47\text{ b}) $$

$$ {}^2R_z = \left[\left({}^2N_a - {}^2B({}^2\kappa)^2\right)\left(\frac{{}^2z'}{{}^2s'}\right) - \frac{{}^2B\,{}^2s''}{({}^2s')^3}\frac{\partial}{\partial\alpha}\left(\frac{{}^2z'}{{}^2s'}\right)\right] + {}^2T\,{}^2\kappa\,{}^2b_z - \left[\frac{{}^2B}{({}^2s')^2}\frac{\partial}{\partial\alpha}\left(\frac{{}^2z'}{{}^2s'}\right)\right]' \qquad (47\text{ c}) $$

$$ {}^2q_x = {}^2s'\left[f_{Hx} - m_p a_{px} - m_i a_{Fx}\right] \qquad (48\text{ a}) $$

$$ {}^2q_y = {}^2s'\left[-w_e + f_{Hy} - m_p a_{py} - m_i a_{Fy}\right] \qquad (48\text{ b}) $$

$$ {}^2q_z = {}^2s'\left[f_{Hz} - m_p a_{pz} - m_i a_{Fz}\right] \qquad (48\text{ c}) $$

The Euler's equation (45) can be written in vectorial form as

$$ \left[\frac{{}^iB}{({}^is')^2}\frac{\partial}{\partial\alpha}\left(\frac{{}^i\vec r'}{{}^is'}\right)\right]'' - \left[\left({}^iN_a - {}^iB({}^i\kappa)^2\right)\frac{{}^i\vec r'}{{}^is'} - {}^iB\left(\frac{{}^is''}{({}^is')^3}\right)\frac{\partial}{\partial\alpha}\left(\frac{{}^i\vec r'}{{}^is'}\right)\right]' - \left[\frac{{}^iT}{{}^is'}\left(\frac{{}^i\vec r'}{{}^is'}\times\frac{\partial}{\partial\alpha}\left(\frac{{}^i\vec r'}{{}^is'}\right)\right)\right]' - {}^i\vec q = 0 \qquad (49) $$

VECTORIAL FORMULATION

To validate equation (49), one has to use the relation between three orthogonal coordinate systems and two moment differential equations to eliminate shear forces. As a result, it is found that the six equilibrium equations are reduced to three equations and can be arranged in vectorial form as equation (49).

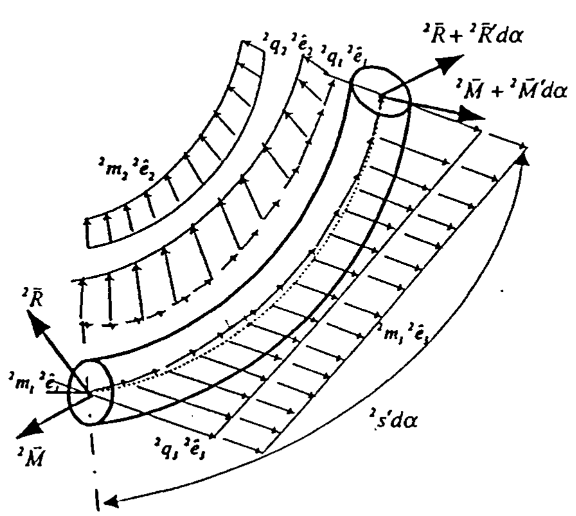

Figure 3. Pipe differential segment.

Figure 3. shows the pipe element of the length $d\,{}^2s$ in displaced state loaded by forces and couples in the cross-sectional principal axes system. Let ${}^2\vec R$ be the vector of an internal force such that ${}^2\vec R = {}^2R_1\,{}^2\hat e_1 + {}^2R_2\,{}^2\hat e_2 + {}^2R_3\,{}^2\hat e_3$ where ${}^2R_1$ is an axial force, ${}^2R_2$ and ${}^2R_3$ are shear forces; let ${}^2\vec M$ be the vector of an internal moment such that ${}^2\vec M = {}^2M_1\,{}^2\hat e_1 + {}^2M_2\,{}^2\hat e_2 + {}^2M_3\,{}^2\hat e_3$ where ${}^2M_1$ is a twisting moment, ${}^2M_2$ and ${}^2M_3$ are bending moments. The vector of an external load, i.e., current and wave force, effective weight, inertial force, is represented by ${}^2\vec q = {}^2q_1\,{}^2\hat e_1 + {}^2q_2\,{}^2\hat e_2 + {}^2q_3\,{}^2\hat e_3$ and the vector of an external distributed moment is represented by ${}^2\vec m = {}^2m_1\,{}^2\hat e_1 + {}^2m_2\,{}^2\hat e_2 + {}^2m_3\,{}^2\hat e_3$. Since the pipe element is in equilibrium, therefore the sum of forces and the sum of moments equal to zero. Hence, the equilibrium equations in the cross-sectional principal axes system are

$$ \frac{{}^2R_1'}{{}^2s'} + {}^2R_3\,{}^2\omega_2 - {}^2R_2\,{}^2\omega_3 = -{}^2q_1 \qquad (50\text{ a}) $$

$$ \frac{{}^2R_2'}{{}^2s'} + {}^2R_1\,{}^2\omega_3 - {}^2R_3\,{}^2\omega_1 = -{}^2q_2 \qquad (50\text{ b}) $$

$$ \frac{{}^2R_3'}{{}^2s'} + {}^2R_2\,{}^2\omega_1 - {}^2R_1\,{}^2\omega_2 = -{}^2q_3 \qquad (50\text{ c}) $$

$$ \frac{{}^2M_1'}{{}^2s'} + {}^2M_3\,{}^2\omega_2 - {}^2M_2\,{}^2\omega_3 = -{}^2m_1 \qquad (50\text{ d}) $$

$$ \frac{{}^2M_2'}{{}^2s'} + {}^2M_1\,{}^2\omega_3 - {}^2M_3\,{}^2\omega_1 = {}^2R_3 - {}^2m_2 \qquad (50\text{ e}) $$

$$ \frac{{}^2M_3'}{{}^2s'} + {}^2M_2\,{}^2\omega_1 - {}^2M_1\,{}^2\omega_2 = -{}^2R_2 - {}^2m_3 \qquad (50\text{ f}) $$

It is worth noticing in this formulation that the external forces are assumed to act on the centerline of the pipe, therefore the distributed external moments are equal to zero.

By coordinate transformation and shear force elimination, the components of internal force vector in fixed Cartesian coordinate can be derived and written in vectorial form as follows

$$ {}^2\vec R = \left[\left({}^iN_a - {}^iB({}^i\kappa)^2\right)\frac{{}^i\vec r'}{{}^is'} - {}^iB\left(\frac{{}^is''}{({}^is')^3}\right)\frac{\partial}{\partial\alpha}\left(\frac{{}^i\vec r'}{{}^is'}\right)\right] - \left[\frac{{}^iB}{({}^is')^2}\frac{\partial}{\partial\alpha}\left(\frac{{}^i\vec r'}{{}^is'}\right)\right]' + \left[\frac{{}^iT}{{}^is'}\left(\frac{{}^i\vec r'}{{}^is'}\times\frac{\partial}{\partial\alpha}\left(\frac{{}^i\vec r'}{{}^is'}\right)\right)\right] \qquad (51) $$

Since, the summation of forces in fixed Cartesian coordinate is

$$ {}^2\vec R' + {}^2\vec q' = 0 \qquad (52) $$

therefore, it is confirmed that exact agreement is achieved among the vectorial formulation and the variational formulation.

APPLICATIONS

The formulation allows users to choose the independent variable α to suit their solutions. The independent variable α can be choosen to be $\{{}^ix, {}^iy, {}^iz, {}^is\}$.

In high-tension pipe, the independent variable α can be used as the water depth ${}^iy$, which is known initially because the displacement function is the one to one function for all points of elastic curve. But

When the pipe is supported by low tension, the displacement function may not be the one to one function. Therefore, using $\alpha = {}^{i}s$ is more suitable because the arc-length parameter is always the one to one function for all points of elastic curves. The in-plane offset ${}^{i}x$ or the out-of-plane offset ${}^{i}z$ can be used as an independent variable when the offset is static. However, the boundary condition is unknown when the offset is dynamic and is not effective when the displacement curve looks like the C-shape or the semi C-shape.

The left superscript i is used to define the state of variable, therefore $i=0$ refers to the analysis performed by using the total Lagrangian descriptor, $i=1$ refers to the updated Lagrangian descriptor, and $i=2$ refers to the Eulerian descriptor.

This formulation is not limited to the extensible marine pipes/risers conveying fluid, but can be readily applied to the other problems of large strain with some modifications in equation (49), for example, three-dimensional elastic rods and marine cables with large displacement, and etc.

CONCLUSIONS

A variational formulation of extensible marine pipe conveying fluid has been presented in three descriptors. The classical mechanics and elastica theory of rod in a three-dimensional space have been used for large strain analysis. The independent variable is used in the formulation for the sake of generality. The formulation has been validated by the equilibrium equations obtained from summation of forces and moments of the pipe element at current state. The advantages of the present formulation are the flexibility of the independent variable, and the application of numerous elastica problems. Moreover, the formulation can be arranged to be the form that suits for many numerical methods such as the finite element method, the shooting method, the Rayleigh-Ritz method etc.

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APPENDIX

Differential Geometry

In Cartesian coordinate, the basic formulas of differential geometry of a space curve are

$$ {}^{i}\kappa = {}^{i}s'\,{}^{i}\theta' , \quad {}^{i}\tau = {}^{i}s'\,{}^{i}\phi' $$

$$ \left({}^{i}s'\right)^2 = \left({}^{i}x'\right)^2 + \left({}^{i}y'\right)^2 + \left({}^{i}z'\right)^2 $$

$$ {}^{i}\theta' = \frac{1}{\left({}^{i}s'\right)^2}\sqrt{\left({}^{i}x''\,{}^{i}y' - {}^{i}x'\,{}^{i}y''\right)^2 + \left({}^{i}y''\,{}^{i}z' - {}^{i}y'\,{}^{i}z''\right)^2 + \left({}^{i}x''\,{}^{i}z' - {}^{i}x'\,{}^{i}z''\right)^2} $$

$$ {}^{i}\phi' = {}^{i}s'\left[\frac{{}^{i}x'\left({}^{i}y''\,{}^{i}z''' - {}^{i}y'''\,{}^{i}z''\right) - {}^{i}y'\left({}^{i}x''\,{}^{i}z''' - {}^{i}x'''\,{}^{i}z''\right) + {}^{i}z'\left({}^{i}x''\,{}^{i}y''' - {}^{i}x'''\,{}^{i}y''\right)}{\left({}^{i}x''\,{}^{i}y' - {}^{i}x'\,{}^{i}y''\right)^2 + \left({}^{i}y''\,{}^{i}z' - {}^{i}y'\,{}^{i}z''\right)^2 + \left({}^{i}x''\,{}^{i}z' - {}^{i}x'\,{}^{i}z''\right)^2}\right] $$

(A1 a-e)

$$ {}^{i}\hat{n} = {}^{i}n_x\hat{i} + {}^{i}n_y\hat{j} + {}^{i}n_z\hat{k} $$

$$ {}^{i}\hat{b} = {}^{i}b_x\hat{i} + {}^{i}b_y\hat{j} + {}^{i}b_z\hat{k} \qquad \text{(A2 a,b)} $$

$$ {}^{i}n_x = \frac{1}{{}^{i}\kappa}\left(\frac{{}^{i}x''}{\left({}^{i}s'\right)^2} - \frac{{}^{i}x'\,{}^{i}s''}{\left({}^{i}s'\right)^3}\right), \quad {}^{i}n_y = \frac{1}{{}^{i}\kappa}\left(\frac{{}^{i}y''}{\left({}^{i}s'\right)^2} - \frac{{}^{i}y'\,{}^{i}s''}{\left({}^{i}s'\right)^3}\right) $$

$$ {}^{i}n_z = \frac{1}{{}^{i}\kappa}\left(\frac{{}^{i}z''}{\left({}^{i}s'\right)^2} - \frac{{}^{i}z'\,{}^{i}s''}{\left({}^{i}s'\right)^3}\right) $$

$$ {}^{i}b_x = \frac{1}{{}^{i}\kappa}\left[\frac{{}^{i}y'\,{}^{i}z'' - {}^{i}z'\,{}^{i}y''}{\left({}^{i}s'\right)^3}\right], \quad {}^{i}b_y = \frac{1}{{}^{i}\kappa}\left[\frac{{}^{i}z'\,{}^{i}x'' - {}^{i}x'\,{}^{i}z''}{\left({}^{i}s'\right)^3}\right] $$

$$ {}^{i}b_z = \frac{1}{{}^{i}\kappa}\left[\frac{{}^{i}x'\,{}^{i}y'' - {}^{i}y'\,{}^{i}x''}{\left({}^{i}s'\right)^3}\right] \qquad \text{(A3 a-f)} $$